Mathematics of Fuzzy SYstems, VERLAG TUV Rheinland Germany (1986)
121-141.

The category of the Fuzzy Models and Lowenheim-Skolem Theorem

by

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1 Introduction

In this paper we propose a utilization of the algebraic approach to non classical logic, as devised by R. Sikorski, H. Rasiowa and other authors, as a first step towards this general foundation of fuzzy set theory. In fact, it turns out that the principal objects of investigation of fuzzy set theory, such as the fuzzy algebras, the fuzzy graphs, the similarity relations, etc. ....., are models (in the sense of the above cited authors) of a first order language (see also [4]). Conversely, we believe that, be providing natural and non "ad hoc" models, fuzzy set theory can give new suggestions and tools to nonclassical logic.

By defining a suitable concept of morphism, we give the class of the non classical models (in this paper "fuzzy models") of L a structure of category F(L). We prove that such a category has direct products. Moreover the concepts of congruence and quotient are defined and the usual homomorphism theorems are proved. Also, suitable notions of reduced product and ultraproduct are given.

In particular we obtain that, for example, the fundamental notions of morphism, congruence, quotient, direct and reduced product, ultraproduct are given for the fuzzy algebras.

All these notions are new.

We prove that quotients, direct products, reduced products, ultraproducts preserve the first order properties.This enables us to extend Lowenheim-Skolem Theorem to a very large class of non classical logics.
Note that the above operations are not generalizations of the analogous

classical ones. For example, the direct product of a family of classical models is, in the category F(L), a boolean valued model.
If L has only a monadic predicate, then the fuzzy sets individuate a full subcategory F of F(L). Such a category is very different from the early defined categories of fuzzy sets. We prove that, in this case, F has initial objects, terminal objects and pullbacks while it has not exponentials and a subobject classifier. Thus F is not a topos.

2 Valuation structures

In the sequel N denotes the natural number set and $M=N\cup\{0\}$. A type for a valuation structure is a family of disjoint sets $\bar{E}$, $\bar{U}$, $\bar{C}_0$, $\bar{C}_1,\ldots,\bar{C}_n$, ... such that $v,f \in \bar{C}_0$, $\wedge \in \bar{C}_2$. Elements of $\bar{E}$ and $\bar{U}$ are called existential and universal quantifiers, respectively, the elements of $\bar{C}_n$, for $n\in M$, n-ary connectives. A valutation structure or generalized algebra (see [12] and [14]) of type $(\bar{E}, \bar{U},(\bar{C}_n)_{n\in M})$ is a pair $\mathcal{V} = (V, I)$, where V is a set and I is a map defined in $\bar{E}\cup\bar{U} \cup (\bigcup_{n\in M} \bar{C}_n)$ such that:

a) to every $\bar{c}\in\bar{C}_n$, I associates an n-ary operation in such a manner that $(V, \wedge, 0,1)$ is a semilattice with zero 0 and unity 1, where $\wedge = I(\;)$, $0 = I(f)$ and $1=I(v)$.

b) to every $\bar{q}\in\bar{E}$ $(\bar{q}\in\bar{U})$ I associates an increasing (decreasing) map $q=I(\bar{q})$ from a class D_q of nonempty subsets of V into V.

Observe that the monotonicity of the quantifiers is not required in [12] and [14]. In our paper such an hypothesis is essential only in the proofs of Proposition 9.1 and Proposition 9.2. The class D_p is called the domain of q. A valuation structure whose domains of quantifiers are equal to $P(V)$ is called complete.
In the sequel we set $\bar{Q}=\bar{E} \cup \bar{U}$, $E= \{I(\bar{q})/\bar{q}\in\bar{E}\}$, $U= \{I(\bar{q})/\bar{q}\in\bar{U}\}$ $Q=E \cup U$, $C_n = \{I(\bar{c})/\bar{c}\in\bar{C}_n\}$, $C= \bigcup C_n$. We denote the valuation structure by (V,C,Q), also.
If (V,I) and (V',I') are valuation structures of the same type, then a homomorphism k from (V,I) to (V', I') is a homomorphism of the algebraic structure (V,C) to (V',C') such that, for every $\bar{q}\in\bar{Q}$ and $X\in D_q$,

$k(X) \varepsilon D_{q'}$ and

(2.1) $k(q(X)) = q'(k(X))$

where $q = I(\bar{q})$ and $q' = I'(\bar{q})$. If k is the identity embedding then (V, I) is called a substructure of (V', I').

A congruence of a valuation structure (V,C,Q) is a congruence ψ of the algebraic structure (V,C) such that, for every $q \varepsilon E$ $(q \varepsilon U)$ and $X,Y \varepsilon D_q$,

(2.2) $[X]_\psi \subseteq [Y]_\psi$ implies $[q(X)]_\psi \leq [q(Y)]_\psi$ $([q(X)]_\psi \geq [q(Y)]_\psi)$

where, for every $Z \subseteq V$, $[Z]_\psi = \{[z]_\psi / z \varepsilon Z\}$. The quotient of (V,C,Q) with respect to ψ is the valuation structure $(V', I') = (V',C',Q')$ such that (V',C') is the quotient of (V,C) and, for every $\bar{q} \varepsilon \bar{Q}$, $D_{q'} = \{[Y]_\psi / Y \varepsilon D_q\}$ and $q'([X]_\psi) = [q(X)]_\psi$.

If $<(V_i, I_i)>_{i \varepsilon I}$ is a family of valuation structures, then the product $(V,I)=(V,C,Q)$ is defined by assuming that (V,C) is the direct product of the family $<(V_i,C_i)>_{i \varepsilon I}$ of algebras.

Moreover, for every $\bar{q} \varepsilon \bar{Q}$, $q = I(\bar{q})$ is defined by setting:

(i) D_q equal to the class of the subsets of V of type $\prod_{i \varepsilon I} X_i$ where, for every $i \varepsilon I$, $X_i \varepsilon D_{q_i}$ $(q_i = I_i(\bar{q}))$

(ii) $q(\prod_{i \varepsilon I} X_i) = < q_i(X_i) >_{i \varepsilon I}$.

Moreover, the complete product is defined by setting:

(iii) $D_q = \{X \subseteq V / \text{ for every } i \varepsilon I \ p_i(X) \varepsilon D_{q_i}\}$

(iv) $q(X) = < q_i(p_i(X)) >_{i \varepsilon I}$.

where $p_i: V \to V_i$ denotes, for every $i \varepsilon I$, the ith-projection. It is immediate that the products and the complete products are valutation structures.

If S is a set and $V = (V,C,Q)$ is a valuation structure, then a n-ary fuzzy relation or V-relation is a map $r: S^n \to V$ from S^n to V. If n=1 then r is called fuzzy subset or V-subset of S. Sometimes, in literature, r is called fuzzy set or V-set on S, [9], [10], [17].

3 Fuzzy models of first order languages.

A generalized first order language L is a first order language (in the classical sense) with a type for a valuation structure. Then a generalized first order language is an ordered system $L = \{(\bar{F}_m)_{m \varepsilon M}, (\bar{R}_m)_{m \varepsilon M},$

$\bar{E}$, $\bar{U}$, $(\bar{C}_m)_{m\varepsilon M}\}$ of disjoint sets. For any mεM, the elements of $\bar{F}_m$ and of $\bar{R}_m$ will be called m-argument fuctors and predicates respectively. Moreover we set $\bar{F}=\bigcup\bar{F}_m$ and $\bar{R}=\bigcup\bar{R}_m$. Terms, open formulas and closed formulas are defined as usual. In particular if α is a formula and x_i a variable, then $\bar{q}x_i\alpha$ is a formula for every $\bar{q}\varepsilon\bar{Q}$. L^* denotes the set of formulas, $\bar{L}$ the set of closed formulas and L_n the set of formulas whose free and bound variables are in $\{x_1,\dots,x_n\}$.
We write $\alpha(x_1,\dots,x_n)$ to denote $\alpha \varepsilon L_n$. A fuzzy model or realization $M=(D,V,I)$ for L is a tern such that D is a set (the domain), V is a set (the valuation set) and I (the interpretation) a map such that V with the restriction of I to the type $\bar{E}\cup\bar{U}\cup(\bigcup\bar{C}_m)$ is a valuation structure $V(M)$. Moreover I associates

- to every $\bar{f} \varepsilon \bar{F}_m$ an m-ary operation $f=I(\bar{f})$ of D
- to every $\bar{r} \varepsilon \bar{R}_m$ an m-ary $V(M)$-relation $r=I(\bar{r})$

We set $F_m = \{I(\bar{f})/\bar{f}\varepsilon\bar{F}_m\}$, $R_m = \{I(\bar{r})/\bar{r}\varepsilon\bar{R}_m\}$, $F=\bigcup F_m$ and $R=\bigcup R_m$. In other words a fuzzy model is determined by a classical algebra $A(M) = (D,F)$, a valuation structure $V = V(M)=(V,C,\dot{Q})$ and a set R of V-relations defined in D.
The valuation of the formulas of L with respect to a fuzzy model M is defined as follows. If $t(x_1,\dots,x_n)$ denotes a term whose free variables are in $\{x_1,\dots,x_n\}$ and if $d_1,\dots,d_n \in D$, then the value $t[d_1,\dots,d_n]$ of t in $d_1,\dots,d_n$ with respect to M is defined as usual. If $\alpha \varepsilon L_n$ then $V(M, \alpha[d_1,\dots,d_n])$, the valuation of α in $d_1,\dots,d_n$ with respect to M is defined by setting:

(i) $V(M,\bar{r}(t_1,\dots,t_p)[d_1,\dots,d_n]) = r(t_1[d_1,\dots,d_n],\dots, t_p[d_1,\dots,d_n])$

(ii) $V(M,\bar{c}(\alpha_1,\dots,\alpha_s)[d_1,\dots,d_n] = c(V(M,\alpha_1[d_1,\dots,d_n]), \dots,V(M,\alpha_s[d_1,\dots,d_n]))$

(iii) $V(M,\bar{q}x_h\beta[d_1,\dots,d_n]) = q(\{V(M,\beta[d_1,\dots,d_{h-1},d,d_h,\dots, d_n])/d\in D\})$

for every $p,s \varepsilon N$, $\bar{r} \varepsilon \bar{R}_p$, $\bar{c} \varepsilon \bar{C}_s$, $t_1,\dots,t_p$ terms,

$\alpha_1,\dots,\alpha_s,\beta \varepsilon L_n$, $\bar{q} \varepsilon \bar{Q}$, $h \varepsilon \{1,\dots,n\}$

Note that if $V(M)$ is not complete, then the valuation can be undefined for some formulas. If this is not the case, then the fuzzy model is cal

led completely valued.
If $\mathcal{V}=(V,C,Q)$ is a valuation structure, then $\mathcal{V}$- system of axioms is any fuzzy subset $\tau : \bar{L} \to V$. A model of τ is any fuzzy model M such that $\mathcal{V}$ is a substructure of $V(M)$ and $V(M,\alpha) = \tau(\alpha)$ for every $\alpha \in \bar{L}$.
A model of a set τ of formulas is a fuzzy model such that $V(M,\alpha)=1$ for every $\alpha \in \tau$. A class of fuzzy models is axiomatizable if it is the class of the models of a suitable set of formulas.
Observe that if the valuation structure is the two elements boolean algebra, then the above definitions give the classical semantics. If the valuation structures are Heyting, Lukasiewicz or modal algebras, then we obtain the semantics for the corrispondent first order logics.
In all this cases the quantifiers are interpreted as the least upper bound and the greatest lower bound operators. But, it is possible to give several other interesting definitions of the quantifiers. For example, we can define a universal quantifier by setting

$$(3.1)\quad q(X) = \begin{cases} 1 & \text{if } X = \{1\} \\ 0 & \text{otehrwise} \end{cases}$$

In the framework of fuzzy set theory interesting definitions of quantifiers are possible that are related with the entropy and energy concepts (see [3], [5]). For example, if in the valuation structure a complementation and an equivalence are defined, then we can define the existential quantifier $q = I(\bar{q})$ by setting

$$q(X) = \sup(\{v \leftrightarrow \sim v / v \in X\}) \qquad X \subseteq V.$$

Then it is reasonable to assume the valuation of the formula $\bar{q}xA$ as the "degree of fuzziness" or "entropy" of the predicate $A(x)$. Obviously we can also express this entropy by the formula $\exists x(A \leftrightarrow \sim A)$
Recall that a fuzzy algebra (see [1], [4], [6], [8], [10], [13]) is a fuzzy subset $f:A \to L$ of an algebra A, with L semilattice, such that

$$(3.2)\quad f(h(d_1,\dots,d_n)) \geq f(d_1) \wedge \dots \wedge f(d_n)$$

for every n-ary operation h of the algebra A and $d_1,\dots,d_n \in A$.
Fuzzy algebra concept is useful to investigate about the lattice of the subalgebras of a given algebra. Indeed a fuzzy subset $f:A \to L$ is a fuzzy algebra if and only if every α-cut $C_\alpha = \{x \in A / f(x) \geq \alpha\}$ is a subalgebra

of A. Then it is possible to identify the fuzzy algebras with suitable families $(C_\alpha)_{\alpha\varepsilon L}$ of subalgebras of A.
The following proposition shows that the fuzzy algebras are fuzzy models of a suitable theory.

Proposition 3.1 The fuzzy algebras are the fuzzy models of the system of formulas
(3.3) $\forall x_1 \ldots \forall x_n\ (\bar{r}(x_1) \wedge \ldots \wedge \bar{r}\ (x_n)) \to \bar{r}(\bar{s}(x_1,\ldots,x_n))\ \bar{s}\varepsilon\bar{F}_n$ with respect to suitable valuation structures. In other words, the fuzzy algebra concept is axiomatizable.

Proof. Let M be a fuzzy model of (3.3) such that, for every $X \varepsilon D_q$ from $q(X) = 1$ it follows $X = \{1\}$ and for every $u,v \ \varepsilon\ V$ from $u \leq v$ it follows that $c(u,v)=1$, where $q=I(\forall)$ and $c=I(\to)$. Then it is immediate that M is a fuzzy algebra.
Conversely, let $f:A \to L$ be a fuzzy algebra and let V be the valuation structure obtained by interpreting $\to$ as the binary operation c defined by

$$c(u,v) = \begin{cases} 1 & \text{if } u \leq v \\ 0 & \text{otherwise} \end{cases}$$

and the quantifier $\forall$ by the map q defined in (3.1). It is immediate that the fuzzy algebra becomes, with respect to such a valuation structure, a model of (3.3).
There are particular fuzzy algebras that are interesting for code theory, namely the free, pure, very pure, left unitary right unitary fuzzy semigroups (see [1], [6], [7], [8]). They are characterized by the property that the corresponding cuts are free, pure, very pure, left unitary, right unitary subsemigroups of the given semigroup, respectively.
The following proposition shows that these classes of fuzzy semigroups are axiomatizable.

Proposition 3.2 The free, pure, very pure, left unitary, right unitary fuzzy semigroups are the models of the system of axioms formed by the formula

(3.4) $\forall x_1\ \forall x_2\ (\bar{r}(x_1) \wedge \bar{r}(x_2) \longrightarrow \bar{r}(x_1 \cdot x_2))$

and, respectively, by the formulas

(3.5) $\forall x_1 \forall x_2 (\bar{r}(x_1.x_2) \wedge \bar{r}(x_2.x_1) \wedge \bar{r}(x_2) \longrightarrow \bar{r}(x_1))$ (free)

(3.6) $\forall x_1 (\bar{r}(x_1^n) \longrightarrow \bar{r}(x_1))$; $n \in N$ (pure)

(3.7) $\forall x_1 \forall x_2 (\bar{r}(x_1 \cdot x_2) \wedge \bar{r}(x_2 \cdot x_1) \longrightarrow \bar{r}(x_1))$ (very pure)

(3.8) $\forall x_1 \forall x_2 (\bar{r}(x_2 \cdot x_1) \wedge \bar{r}(x_2) \longrightarrow \bar{r}(x_1))$ (left unitary)

(3.9) $\forall x_1 \forall x_2 (\bar{r}(x_1 \cdot x_2) \wedge \bar{r}(x_2) \longrightarrow \bar{r}(x_1))$ (right unitary)

Proof. As in Proposition 3.1.

4 The category of the fuzzy models.

Now we define the category $F(L)$ of the fuzzy models of the language L. The objects of this category are the fuzzy models of L, a morphism from $M = (D,V,I)$ into $M'=(D',V',I')$ is a pair (h,k) of homomorphisms from $A(M)$ to $A(M')$ and from $V(M)$ to $V(M')$, respectively, such that the following diagram commutes

(4.1)
$$\begin{array}{ccc} D & \xrightarrow{\ h\ } & D'^n \\ r \downarrow & & \downarrow r' \\ V & \xrightarrow{\ k\ } & V' \end{array}$$

for every $\bar{r} \in R_n$, where $r = I(\bar{r})$, $r' = I'(\bar{r})$ and $h(d_1,\dots,d_n)$ is $(h(d_1),\dots,h(d_n))$ for every $(d_1,\dots,d_n) \in D^n$. The product of two morphisms (h,k) and (h',k') is the morphism (hh',kk').

A morphism (h,k) such that h and k are both monomorphisms, epimorphisms or isomorphisms is called monomorphism, epimorphism or isomorphism, respectively. Note that if the equality $=$ is an element of R_2 and it is classically interpreted, then the commutativity of (4.1) implies that h is injective. If h and k are the identity embedding, then M is called a submodel of M'. A morphism is elementary if

(4.2) $V(M', \alpha[h(d_1),\dots,h(d_n)]) = k(V(M, \alpha[d_1,\dots,d_n]))$ for every $\alpha \in L_n$ and $d_1,\dots,d_n \in D$. If (h,k) is an elementary monomorphism, then M' is called an elementary extension of M. If h and k are the identity embedding then M is called an elementary submodel of M'.

The following proposition gives a condition in order that a morphism is elementary.

Proposition 4.1. Let M and M' be two completely valued fuzzy models and (h,k) a morphism from M to M'.

Moreover, assume that for every formula $\beta \in L_n$, $\bar{q} \in \bar{Q}$, $i \in \{1,\dots,n\}$ and $d_1,\dots,d_n \in D$

(4.3) $V(M', \bar{q}x_i\ \beta[h(d_1),\dots,h(d_n)]) = q'(\{V(M', \beta[h(d_1),\dots$
$\dots, h(d_{i-1}), h(d), h(d_{i+1}),\dots,h(d_n)])/d \in D\})$

Then (h,k) is elementary.

Proof. We prove (4.2) by induction on the complexity of α. Suppose that α is atomic, i.e. of type $\bar{r}(t_1,\dots,t_p)$ with $\bar{r} \in \bar{R}_p$ and $t_1,\dots,t_p$ terms. Since h is a homomorphism, we have, for every $d_1,\dots,d_n \in D$ and $i \in \{1,\dots,p\}$, $t_i'(h(d_1),\dots,h(d_n)) = h(t_i(d_1,\dots,d_n))$, with obvious meaning of the symbols. Then, by the commutativity of (4.1), $V(M', \bar{r}(t_1,\dots,t_p)[h(d_1),\dots,h(d_n)]) =$
$r'(h(t_1[d_1,\dots,d_n]),\dots,h(t_p[d_1,\dots,d_n])) =$
$k(r(t_1[d_1,\dots,d_n],\dots,t_p[d_1,\dots,d_n]))$
$k(V(M, \bar{r}(t_1,\dots,t_p)[d_1,\dots,d_n])$

Suppose that $\alpha = \bar{c}(\alpha_1,\dots,\alpha_p)$, then
$V(M', \alpha[h(d_1),\dots,h(d_n)]) = c'(V(M', \alpha_1[h(d_1),\dots,h(d_n)]),\dots$
$\dots, V(M', \alpha_p[h(d_1),\dots,h(d_n)])) = c'(k(V(M, \alpha_1[d_1,\dots,d_n]),\dots$
$\dots, k(V(M, \alpha_p[d_1,\dots,d_n])) = k(c(V(M, \alpha_1[d_1,\dots,d_n])\dots$
$\dots, V(M, \alpha_p[d_1,\dots,d_n]))) = k(V(M, \alpha[d_1,\dots,d_n]))$.

Finally, suppose that $\alpha = qx_i\ \beta$, then by (4.3) and the inductive hypothesis we have
$k(V(M, \bar{q}x_i\ \beta[d_1,\dots,d_n]) = k(q(\{V(M, \beta[d_1,\dots,d_{i-1},d,d_{i+1},\dots$
$\dots, d_n])/d \in D\})) = q'(\{k(V(M, \beta[d_1,\dots,d_{i-1},d,d_{i+1},\dots,d_n])/d \in D\}) =$
$q'(\{V(M', \beta[h(d_1),\dots,h(d_{i-1}),h(d), h(d_{i+1}),\dots,h(d_n)])/d \in D\}) =$
$V(M', \bar{q}x_i\ \beta[h(d_1),\dots,h(d_n)])$

This completes the proof.

Proposition 4.2 Let M be a completely valued fuzzy model and (h,k) a morphism from M to a fuzzy model M' with h surjective. Then M' is completely valued and (h,k) is elementary.

Proof. It suffices to repeat the proof of Proposition 4.1. The only difference is that we have to prove that M' is completely valued. We confine ourself to observe that if $\{V(M, \beta[d_1,\dots,d_{i-1},d,d_{i+1},\dots,d_n])/d \in D\}$ is in D_q, then $k(\{V(M, \beta[d_1,\dots,d_{i-1},d, d_{i+1},\dots,d_n])/d \in D\})$

is in $D_{q'}$. Moreover, by inductive hypothesis,
$k(\{(V(M, \beta[d_1,\dots,d_{i-1},d,d_{i+1},\dots,d_n])/d \in D\}) =$
$\{V(M', \alpha[h(d_1),\dots,h(d_{i-1}),h(d),h(d_{i+1}),\dots,h(d_n)])/d \in D\})$
$\{V(M', \alpha[h(d_1),\dots,h(d_{i-1}),d',h(d_{i+1}),\dots,h(d_n)])/d' \in D'\}$
This compeltes the proof.

5 Direct products

In order to show that the category F(L) of the fuzzy models has direct products and to give an interesting example of elementary extension, we recall some definitions given in [17]. If $\langle M_i \rangle_{i\in I}$ is a family of fuzzy models of L , $M_i = (D_i, V_i, I_i)$, then we call product (complete product) of the family the fuzzy model $M = (D,V,I)$ such that $A(M)$ is the direct product of $A(M_i)_{i\in I}$, $V(M)$ is the product (the complete product) of $V(M_i)_{i\in I}$ and $r(d_1,\dots,d_n) = \langle r_i(d^i_1,\dots,d^i_n)\rangle_{i\in I}$ for every $d_1 = \langle d^1_i\rangle_{i\in I},\dots,d_n = \langle d^n_i\rangle_{i\in I}$ elements of $D = \prod D_i$, $\bar{r} \in \bar{R}$ and $n \in M$, where $r = I(\bar{r})$, $r_i = I_i(\bar{r})$. If all the components M_i are equal, then we call M power (complete power, respectively). We denote a product and a complete product by $\prod M_i$ and $\bar{\prod} M_i$, respectively. Observe that the product (and the complete product) of a family of classical models is not a classical model and it is not with classical equality. Such products are boolean valued structures.

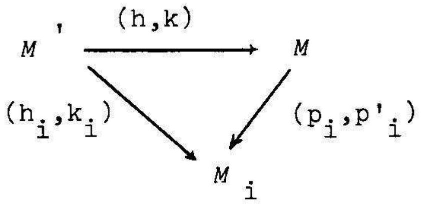
Proposition 5.1 Let $\langle M_i \rangle_{i\in I}$ be a family of fuzzy models of L, $M = \bar{\prod} M_i$ the complete product and $p_i : \prod D_i \longrightarrow D_i$, $p'_i : \prod V_i \longrightarrow V_i$ the ith-projection maps. Then M is the direct product in the category F (L) with respect to the family $\langle (p_i, p'_i)\rangle_{i\in I}$ of morphisms.

Proof. To prove that (p_j, p'_j) is a morphism, it suffices to observe that, by definition, for every $X \in D_q$

$$p'_j(q(X)) = p'_j(\langle q'_j(p'_j(X))\rangle_{i\in I}) = q_j(p'_j(X))$$

and that, if $d_1,\dots,d_n$ are elements of D,
$p'_j(r(d_1,\dots,d_n)) = p'_j(\langle r_i(d^1_i,\dots,d^n_i)\rangle_{i\in I}) = r_j(d^1_j,\dots,d^n_j) = r_j(p_j(d_1),\dots,p_j(d_n))$.
Let $M' = (D', V', I')$ be any fuzzy model and, for every $i \in I$, let (h_i, k_i)

be a morphism from M' to M_i. We have to prove that a morphism (h,k) exists such that the following diagram commutes

(5.1) 

Now, define $h:D'\longrightarrow D$ and $k:V'\longrightarrow V$ by setting $h(d)=\langle h_i(d)\rangle_{i\varepsilon I}$, $k(v)=\langle k_i(v)\rangle_{i\varepsilon I}$ for every $d\varepsilon D'$ and $v\varepsilon V'$.
It is obvious that h is a homomorphism from (M') to $A(M)$. To prove that k is a homomorphism from $V(M')$ to $V(M)$, it is sufficient to observe that if $X\ \varepsilon\ D_{q'}$ then $q(k(X))=\langle q_i(p'_i(k(X)))\rangle_{i\varepsilon I}=\langle q_i(k_i(X))\rangle_{i\varepsilon I}=\langle k(q'(X))\rangle_{i\varepsilon I}=k(q'(X))$.
To prove that (h,k) is a morphism observe that:
$k(r'(d_1,\dots,d_n))=\langle k_i(r'(d_1,\dots,d_n))\rangle_{i\varepsilon I}=$
$=\langle r_i(h_i(d_1),\dots,h_i(d_n))\rangle_{i\varepsilon I}=r(h(d_1),\dots,h(d_n))$.
The commutativity of (5.1) is immediate.

The following proposition is proved in [17]. It shows that, differently from the classical case, direct products of fuzzy models preserve first order properties.

Proposition 5.2 If $\langle M_i\rangle_{i\varepsilon I}$ is a family of completely valued fuzzy models, then its product (complete product) M is completely valued and
(5.2) $V(M,\alpha[d_1,\dots,d_n])=\langle V(M_i,[d_1^i,\dots,d_n^i])\rangle_{i\varepsilon I}$
for every formula $\alpha\varepsilon L_n$ and $d_1=\langle d_i^1\rangle_{i\varepsilon I},\dots\dots,d_n=\langle d_i^n\rangle_{i\varepsilon I}$ elements of D.

The following proposition shows that the subcategories of the fuzzy algebras and of the free, pure, very pure, left and right unitary fuzzy semigroups have direct products.

Proposition 5.3 The product (complete product) of a family of fuzzy algebras is a fuzzy algebra. The same holds for the free, pure, very pure left unitary, right unitary fuzzy semigroups.

Proof. It follows from Proposition 3.1, Proposition 3.2 and Proposition 5.2.

Let I be a set and $\mathcal{M} = (D, V, \mathcal{I})$ a fuzzy model. Then in the following proposition we identify the elements of D (of V) with the corresponding constant maps from I to D (to V respectively).

Proposition 5.4 Let $\mathcal{M}$ be a completely valued fuzzy model, I a set and $\mathcal{M}' = \mathcal{M}^I$ the power (the complete power) of $\mathcal{M}$ with index set I. Then $\mathcal{M}'$ is completely valued and

$$V(\mathcal{M}', \alpha[d_1, \ldots, d_n]) = V(\mathcal{M}, \alpha[d_1, \ldots, d_n]) \tag{5.3}$$

for every formula $\alpha \in \mathcal{L}_n$ and $d_1, \ldots, d_n \in D$ Thus every complete power of $\mathcal{M}$ is an elementary extension of M .

Proof. Obvious.

Observe that the power $\mathcal{M}'$ is not an extension of $\mathcal{M}$. Indeed, if k is the natural embedding of V into V^I, then from $X \in D_q$ we cannot infer that $k(X) \in D_{q'}$.

6 The category of the fuzzy sets.

Suppose that $\mathcal{L}$ has one monadic predicate and that the connectives are $\wedge$, $\vee$, 0, 1. Then the class of the fuzzy models of $\mathcal{L}$ whose valuation structure is a lattice coincides with the class of the fuzzy sets. We denote by F the full subcategory of $F(\mathcal{L})$ individuated by such a class. F is different from the analogous categories of fuzzy sets already known in literature (for references see [11] and [16]). For example F differs from Goguen's category F(L) since in F(L) one refers to the prefixed lattice L only. Moreover, Goguen's definition of morphism h from $a:A \to L$ into $b:B \to L$ requires that $f(x) \leq g(h(x))$ and not just that $f(x) = g(h(x))$.

Also, radical differences exist with respect to Eytan's category Fuz (L) and Higgs topos Sh(L). For example, in such categories morphisms are fuzzy maps. All these categories are not equivalent.

In the sequel we call classical an object of F whose lattice is the one element degenere lattice $\{1\}$. Then by identifying every set X with the classical fuzzy set $f:X \to \{1\}$, the category Set can be considered as a full subcategory of F. Note that there is no morphism from a clas-

sical object into a nonclassical object. Indeed, there is no homomorphism from {1} into a nondegenere lattice L.
Instead, if $b:B \to \{1\}$ is classical and $a:A \longrightarrow L$ is any fuzzy set, then every map $h:A \quad B$ defines a unique morphism from a to b.
The following proposition gives some further informations on the category F.

Proposition 6.1 The category F has direct products, an initial object 0, a terminal object 1 and pullbacks. Moreover F neither has exponentials nor subobject classifiers and therefore is not a topos.
Proof. Since the direct product of a family of lattices is a lattice, from Proposition 5.1 it follows that F has products. The fuzzy set $f:\emptyset \to \{0,1\}$ is an initial object, a fuzzy set of type $f:\{a\} \to \{1\}$ is a terminal object.
Let $a:A \to L_a$, $b:B \longrightarrow L_b$ and $c:C \to L_c$ be fuzzy sets and (h,k), (h',k') two morphisms from a to c and from b to c, respectively. Then it is matter of routine to prove that, if $S=\{(x,y) \varepsilon A x B / h(x)=h'(y)\}$, $L_s=\{(x,y) \ L_s x L_b / k(x)=k'(y)\}$ and $s((x,y))=(a(x),b(y))$, then $s:S \to L_s$ is the pullback of (h,k) and (h',k').
Assume, by absurd, that a subobject classifier Ω exists.
Since a classifier has elements $t:1 \longrightarrow \Omega$, Ω is a classical fuzzy set.
Now, Let $a:A \to [0,1]$ be a fuzzy set such that a(A) is the set U of the rational numbers of $[0,1]$. Moreover denote by $\bar{a}:A \longrightarrow U$ the fuzzy set defined by setting $\bar{a}(x)=a(x)$ for every $x \varepsilon A$. Then the identical embeddings define a monomorphism m from $\bar{a}$ to a. We claim that there is no morphism (h,k) such that the diagram

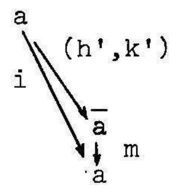
(6.1)
$$\begin{array}{ccc} \bar{a} & \longrightarrow & 1 \\ m\downarrow & & \downarrow t \\ a & \xrightarrow[(h,k)]{} & \Omega \end{array}$$

is a pullback. Indeed if (6.1) is a pullback, since Ω is classical and (6.1) commutes, we have also that

(6.2)
$$\begin{array}{ccc} a & \longrightarrow & 1 \\ i\downarrow & & \downarrow t \\ a & \xrightarrow[(h,k)]{} & \Omega \end{array}$$

commutes, where i is the identical map. Then a morphism (h',k') from a to $\bar{a}$ exists such that

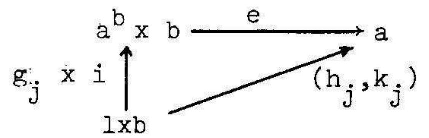
(6.3)



commutes. It follows that h' is the identical mapping. Moreover, since k'(a(x))=a(h'(x))=a(x)=a(x), the morphism k':[0,1]→ U is surjective. This contradicts the fact that there is no increasing map from [0,1] on U. Thus (6.1) is not a pullback and therefore F has no classifiers.

Assume that F has exponentials, let $a:A \longrightarrow L_a$ be nonclassical and $b:B \to L_b$ a fuzzy set such that two morphisms (h_1,k_1) and (h_2,k_2) from 1xb to a exists with $k_1 \neq k_2$. Then an "evaluation" morphism e=(h,k) exists such that

(6.4)



commutes for a suitable morphism $g_j:1 \longrightarrow a^b$. This means that a^b is classical and that

$k_1(1,v)=(1,k(i(v)))=(1,k(v))$ and $k_2(1,v)=(1,k(i(v)))=(1,k(v))$ for every $v \in L_b$. This contradicts the hypothesis $k_1 \neq k_2$.

7. Congrueces and quotients

Let M be a fuzzy model, θ a congruence of $A(M)$ and ψ a congruence of $V(M)$. Moreover assume that, for every $r \in R_n$ and $d_1,\dots,d_n \in D$

(7.1) $d_1 \equiv_\theta d'_1,\dots,d_n \equiv_\theta d'_n \Longrightarrow r(d_1,\dots,d_n) \equiv_\psi r(d'_1,\dots,d_n)$ Then the pair (θ,ψ) is called a congruence of M. The quotient $M' = M/(\theta,\psi)$ of M is defined by setting $A(M')=A(M)/\theta$, $V(M')=V(M)/\psi$ and, every $\bar{r} \in \bar{R}_n$ and $d_1,\dots,d_n \in D$

(7.2) $r'([d_1]_\theta,\dots,[d_n]_\theta) = [r(d_1,\dots,d_n)]_\psi$

where $r' = \mathcal{I}'(\bar{r})$. The above defined concepts of congruence and quotient are not generalizations of the usual ones. For example, let r be the classical interpretation of the equality = in a classical model. Moreover, assume that ψ is nondegenere. Then Θ coincides with the classical equality.

As in the classical case, we can prove the homomorphism theorems.

Proposition 7.1 If (h,k) is a morphism from a fuzzy model $\mathcal{M}$ to a fuzzy model $\mathcal{M}'$, then the pair

$\Theta = \{(d,d') \in D^2 / h(d)=h(d')\}$, $\psi = \{(v,v') \in V^2 / k(v)=k(v')\}$

defines a congruence of $\mathcal{M}$, the kern of (h,k).

Proof. It is obvious that Θ is a congruence of $A(\mathcal{M})$. To prove that ψ is a congruence of $V(\mathcal{M})$, let $\bar{q} \in \bar{E}$ let X and Y be elements of D_q and assume that $[X]_\psi \subseteq [Y]_\psi$. Then it is immediate that $k(X) \subseteq k(Y)$. It follows that $k(q(X))=q'(k(X)) \leq q'(k(Y))=k(q(Y))$ and therefore that $k(q(X) \wedge q(Y))=k(q(X)) \wedge k(q(Y))=k(q(X))$. This proves that $[q(X) \wedge q(Y)] = [q(X)]_\psi$ and therefore that $[q(X)]_\psi \wedge [q(Y)]_\psi = [q(X) \wedge q(Y)]_\psi = [q(X)]_\psi$. In conclusion $[q(X)]_\psi \leq [q(Y)]_\psi$. In a similar manner one proceeds if $\bar{q} \in \bar{U}$. This proves that ψ is a congruence of $V(\mathcal{M})$. To prove that (Θ,ψ) is a congruence of $\mathcal{M}$ let $\bar{r} \in \bar{R}$, $d_1,\ldots,d_n,d'_1,\ldots,d'_n \in D$ and assume that $d_1 \equiv_\Theta d'_1,\ldots,d_n \equiv_\Theta d'_n$. Then $h(d_1)=h(d'_1),\ldots, h(d_n)=h(d'_n)$ and therefore, by the commutativity, $k(r(d_1,\ldots,d_n))=r'(h(d_1),\ldots,h(d_n))=r'(h(d'_1),\ldots,h(d'_n))=k(r(d'_1,\ldots,d'_n))$. It follows that $r(d_1,\ldots,d_n) \equiv_\psi r(d'_1,\ldots,d'_n)$.

Proposition 7.2 Let (Θ,ψ) be a congruence of the fuzzy model $\mathcal{M}$ and the corresponding quotient. Then the maps h and k defined by setting, for every $d \in D$ and $v \in V$ $h(d)=[d]_\Theta$, $k(v)=[v]_\psi$ constitute an epimorphism (h,k) whose kern is (Θ,ψ).

Proof. Let $\bar{q} \in \bar{Q}$, $q=\mathcal{I}(\bar{q})$, $q'=\mathcal{I}'(\bar{q})$ and $X \in D_q$, then from the definition of quotient it follows that k(X) is in $D_{q'}$. Moreover $q'(k(X))=q'([X]_\psi)=[q(X)]_\psi = k(q(X))$. Finally, if $\bar{r} \in \bar{R}_n$ and $d_1,\ldots,d_n \in D$, then $k(r(d_1,\ldots,d_n))=[r(d_1,\ldots,d_n)]_\psi = r'([d_1]_\Theta,\ldots,[d_n]_\Theta)$.

Proposition 7.3 Let $\mathcal{M}$ be a completely valued fuzzy model and (Θ,ψ) a congruence of M . Then the relative quotient M' is completely valued and

(7.3) $V(M', \alpha[[d_1]_\theta, \ldots, [d_n]_\theta]) = [V(M, \alpha[d_1, \ldots, d_n]]_\psi$

for any formula $\alpha \in L_n$ and $d_1, \ldots, d_n \in D$.

Proof. It follows from Proposition 4.2 and Proposition 7.2.

Proposition 7.4 Every quotient of a fuzzy algebra (of a free, pure, very pure, right unitary, left unitary fuzzy semigroup) is a fuzzy algebra (a free, pure, very pure, right unitary, left unitary semigroup, respectively).

Proof. It follows from Proposition 7.3, Proposition 3.1 and Proposition 3.2.

Proposition 7.4 shows that in the categories of the fuzzy algebras and of the free, pure, very pure, right unitary, left unitary fuzzy semigroups the concepts of congruence and quotient are definable in a suitable manner.

8 Reduced Products and Ultraproducts.

Let L_A be the classical first order language with equality whose function symbol set is $\bar{C}$ and with a unique monadic predicate A. Then we say that a quantifier q of a fuzzy model M is <u>defined by a formula</u> $\alpha_q(x_1)$ of L_A if, for every $X \in D_q$,

(8.1) $(V, X) \models \alpha_q[v] \Longleftrightarrow v = q(X)$

where (V, X) is the model of L_A with domain V interpreting $\bar{c} \in \bar{C}$ by $I(\bar{c})$ and the predicate A by X. It is easy to see that in the known logics (classical, intuitionistic, many valued, modals, etc...) the quantifiers are defined by suitable formulas. For example the existential quantifier is defined by the formula

$[\forall x_2(A(x_2) \to x_1 \geqslant x_2] \wedge [\forall x_2(A(x_2) \to x_3 > x_2) \to x_3 > x_1]$.

Even the quantifiers given in Section 3 are definable.

If I is a set, then a <u>d-filter</u> is a pair (F, F') of filters on I such that $F \subseteq F'$ and F' is a ultrafilter. If F is a ultrafilter, i.e. if $F = F'$, then (F, F') is called <u>ultrafilter</u> and it is deno ted by $\mathcal{F}$.

Proposition 8.1 Let $\langle M_i \rangle_{i\varepsilon I}$ be a family of fuzzy models such that for every $\bar{q}\varepsilon\bar{Q}$ there exists a formula $\alpha_{\bar{q}}$ of L_A defining $q_i = I_i(\bar{q})$ for every $i\varepsilon I$. Moreover let ($\mathcal{F}$, $\mathcal{F}'$) be a d-filter and Θ and ψ the relations defined by setting, for every $d = \langle d_i \rangle_{i\varepsilon I}$, $d' = \langle d_i' \rangle_{i\varepsilon I}$ in $\prod D_i$ and $v = \langle v_i \rangle_{i\varepsilon I}$, $v' = \langle v'_i \rangle_{i\varepsilon I}$ in $\prod V_i$,

(8.2) $d \equiv_\Theta d' \iff \{i\varepsilon I / d_i = d_i'\} \varepsilon \mathcal{F}$

(8.3) $v \equiv_\psi v' \iff \{i\varepsilon I / v_i = v_i'\} \varepsilon \mathcal{F}'$

Then (Θ , ψ) is a congruence of the product $M = (D, V, I)$ such that in the corresponding quotient the quantifiers are defined by the same formulas.

Proof. The hard part of the proof is to prove that if $\bar{q}\varepsilon\bar{E}$ ($\bar{q}\varepsilon\bar{U}$) and $X = [\prod X_i]_\psi \subseteq Y = [\prod Y_i]_\psi$ then $[\langle q_i(X_i)\rangle_{i\varepsilon I}]_\psi \leqslant [\langle q_i(Y_i)\rangle_{i\varepsilon I}]$ (respectively $[\langle q_i(X_i)\rangle_{i\varepsilon I}]_\psi \geqslant [\langle q_i(Y_i)\rangle_{i\varepsilon I}]_\psi$) where $\langle X_i \rangle_{i\varepsilon I}$ and $\langle Y_i \rangle_{i\varepsilon I}$ are two families such that $X_i, Y_i \varepsilon D_{q_i}$ for every $i\varepsilon I$. Now, let L_{AB} be an extension of L_A obtained by adding a new monadic predicate B. We set $\alpha_{\bar{q}} = \alpha_{\bar{q}}(A)$ to enphatize the dependence of $\alpha_{\bar{q}}$ from A and denote with $_{\bar{q}}(B)$ the formula of L_{AB} obtained by substituting in α_q every occurrence of A by B. Moreover, to every structure (V_i, C_i) we can associate a model (V_i, C_i, X_i, Y_i) of the language L_{AB} by interpreting c_i by $c_i = I(\bar{c}_i)$, A by X_i and B by Y_i. Then it is immediate that $(V_i, C_i, X_i, Y_i) \models \alpha_{\bar{q}}(A)\ [q(X_i)]$ and $(V_i, C_i, X_i, Y_i) \models \alpha_{\bar{q}}(B)\ [q(Y_i)]$
Since q_i is increasing, we have, for every $i\varepsilon I$, $(V_i, C_i, X_i, Y_i) \models ((\forall x_1(A(x_1) \to B(x_1))) \to x_1 \leqslant x_2)\ [q(X_i), q(Y_i)]$
Now, if (V, I) is the ultraproduct of the family $\langle (V_i, C_i, X_i, Y_i) \rangle_{i\varepsilon I}$, it is immediate that

$(V, I) \models \alpha_{\bar{q}}(A)\ [a]$ where $a = [\langle q(X_i)\rangle_{i\varepsilon I}]_\psi$

$(V, I) \models \alpha_{\bar{q}}(B)\ [b]$ where $b = [\langle q(Y_i)\rangle_{i\varepsilon I}]_\psi$

Moreover

(8.4) $(V, I) \models ((\forall x_1(A(x_1) \to B(x_1))) \to x_1 \leqslant x_2)\ [a, b]$

Now, since $I(A) = X$ and $I(B) = Y$, from $X \subseteq Y$ and (7.4) it follows that $a \leqslant b$. In the same way we proceed if $\bar{q}\varepsilon\bar{U}$.

The quotient of the product of a family of fuzzy models via the congruence associated to a d-filter (a ultrafilter) is called reduced product (ultraproduct, respectively). If all the components M_i of the family are equal, then reduced products and ultraproducts are called reduced powers and ultrapowers respectively.

Proposition 8.2 Let M be the reduced product of a family $\langle M_i \rangle_{i\varepsilon I}$ of completely valued fuzzy models with respect to a d-filter (F, F'). Then M is completely valued and, for every formula $\alpha \varepsilon L_n$ and $d_1,\ldots,d_n \varepsilon D$,

(8.5) $V(M,\alpha[[d_1]_\Theta,\ldots,[d_n]_\Theta])=[\langle V(M_i, \alpha[d_1^i,\ldots,d_n^i])\rangle_{i\varepsilon I}]_\psi$

where Θ and Ψ are the congruences associated to (F , F')

and $d_1 = \langle d_1^i\rangle_{i\varepsilon I},\ldots,d_n = \langle d_n^i\rangle_{i\varepsilon I}$

Proof. Since M is a quotient of the product $\prod M_i$, from Proposition 5.2 and Proposition 8.3 it follows that M is completely valued. Moreover, by the same propositions,

$V(M,\alpha[[d_1]_\Theta,\ldots,[d_n]_\Theta]) =[V(\prod M_i, \alpha[d_1,\ldots,d_n])]_\psi =$

$[\langle V(M_i, \alpha[d_1^i,\ldots,d_n^i])\rangle_{i\varepsilon I}]_\psi$

Observe that a ultraproduct of a family of fuzzy models with classical equality is with classical equality, too. Moreover a ultraproduct of a family of classical models is a classical model. Assume that $V(M_i) = V$ for every $i\varepsilon I$ and that a suitable topology is defined in V. Then we obtain Chang and Keisler's definition of ultraproduct [2] by identifying two elements of the ultrapower V^I/F with the same limit. In the following proposition we identify every element $d\varepsilon D(v\varepsilon V)$ with the class of equivalence of the map constantly equal to d(to v, respectively).

Proposition 8.3 Let M be a completely valued fuzzy model with definable quantifiers, I a set and (F, F') a d-filter on I. Then the relative reduced power M' is an elementary extension of M and, for every $\alpha \varepsilon L_n$, $d_1,\ldots,d_n \varepsilon D$.

(8.6) $V(M', \alpha[d_1,\ldots,d_n]) = V(M, \alpha[d_1,\ldots,d_n])$

Proof. Obvious.

9 A Lowenheim-Skolem type theorem.

The <u>cardinal</u> $|M|$ <u>of a fuzzy model</u> M is the cardinal of the relative domain. The <u>quantifier cardinal</u> $p(M)$ is the minimum cardinal γ such that, for every $q\varepsilon Q$, $X\in D_q$, there exists $Y\varepsilon D_q$ such that $Y\subseteq X$, $|Y|\leq\gamma$ and $q(Y)=q(X)$. Obviously $p(M)\leq|V|$. If $Q=\{\inf., \sup.\}$ and $V(M)$ is well ordered, then $p(M)=1$. If $V(M)$ is the real number interval $[0,1]$ then $p(M)=\omega$. The cardinal $|L|$ is the cardinal of L^*.

Proposition 9.1 Let M' be a completely valued fuzzy model. Then, for every subset X of the domain D' of M', there exists an elementary submodel $M=(D,V,I)$ of M' such that $X\subseteq D$, $V(M)=V(M')$ and

(9.1) $|X|\leq|M|\leq| \; |X|+|L|+q(M')$

Proof. Let $\beta\varepsilon L_n$, $\bar q\varepsilon\bar Q$, $d_1,\ldots,d_n\varepsilon D'$ and $i\ \varepsilon\ \{1,\ldots.,n\}$.
Moreover, set, for every $d\varepsilon D'$,

$$f(d) = V(M', \beta[d_1,\ldots,d_{i-1},d,d_{i+1},\ldots,d_n])$$

Since M' is completely valued $\{f(d)\varepsilon V'/d\ \varepsilon D_{q'}\}\ \varepsilon\ D$ and there exists a subset $S(\beta,\bar q,i,d_1,\ldots,d_n)\varepsilon D_q$, such that

(i) $S(\beta,\bar q,i,d_1,\ldots.,d_n)\subseteq\{f(d)\ \ V'/d\varepsilon D'\}$

(ii) $|S(\beta,\bar q,i,d_1,\ldots.,d_n)|\leq p(M')$

(iii) $V(M', \bar q x_i\ \beta[d_1,\ldots,d_n])=q'(S(\beta,\bar q,i,d_1,\ldots.,d_n))$.

Also, it follows that there exists a subset $X(\beta,q,i,d_1,\ldots.,d_n)$ of D' such that $f(X(\beta,\bar q,i,d_1,\ldots.,d_n)) = S(\beta,\bar q,i,d_1,\ldots.,d_n)$ and $|X(\beta,\bar q,i,d_1,\ldots,d_n)|\leq p(M')$.
We set, for every subset Y of D', C(Y) equal to the subalgebra of $A(M')$ generated by Y and the set $\bigcup\{X(\beta,\bar q,i,d_1,\ldots.,d_n)/n\varepsilon N,\beta\varepsilon L_n,\bar q\varepsilon\bar Q,\ i\varepsilon\{1,\ldots,n\},\ d_1,\ldots,\ d_n\varepsilon\ Y\}$.
Obviously, $|C(Y)|\leq|Y|+|L|+p(M')$. Moreover, set $X_0=X, X_n=C(X_{n-1})$, $D=\bigcup X_n$. It is immediate that D is the domain of a sub algebra of $A(M')$ such that $|D|\leq|X|+|L|+p(M')$. Let $M=(D,V,I)$ be the submodel of M' such that $A(M)$ is the subalgebra of $A(M')$ with

domain D, V(M) = V(M') and, for every $\bar{r}\varepsilon\bar{R}_n$, $r = I(\bar{r})$ is the restriction of $r' = I'(\bar{r})$ to D^n. In order to prove that M is an elementary submodel of M', we have to prove, by Proposition 4.1, that

(9.2) $V(M', \bar{q}x_i\beta[d_1,\dots,d_n]) = q'(\{ V(M', \beta[d_1,\dots,d_{i-1},d,d_{i+1},\dots, d_n])/d\varepsilon D\})$

for every $n\varepsilon N$, $\beta\varepsilon L_n$, $\bar{q}\varepsilon\bar{Q}$, $i\ \varepsilon\{1,\dots,n\}$, $d_1,\dots,d_n\varepsilon$ D.

Now, if $d_1,\dots,d_n\varepsilon$ D then there exists $j\varepsilon M$ such that $d_1,\dots,d_n \in X_j$.

Moreover,

$V(M', qx_i\ \beta[d_1,\dots,d_n]) = q'(S(\beta, \bar{q}, i, d_1,\dots,d_n)) =$

$q'(\{ f(d)\varepsilon V'/d\varepsilon X(\beta, \bar{q}, i, d_1,\dots,d_n)\})$

where, by construction, $X(\beta,\bar{q},i,d_1,\dots,d_n)$ is a subset of X_{j+1} and, therefore, of D. Suppose that $\bar{q}\varepsilon\bar{E}$ then q' is increasing and we have that

$V(M', \bar{q}x_i\ \beta[d_1,\dots,d_n]) = q'(\{ f(d)\varepsilon V'/d\ \varepsilon D'\}) \geqslant$

$q'(\{ f(d)\varepsilon V'/d\ \varepsilon D\}) \geqslant q'(\{ f(d)\varepsilon V'/d\ \varepsilon X_{j+1}\}) \geqslant$

$q'(\{ f(d)\varepsilon V'/d\varepsilon X(\beta,\bar{q},i,d_1,\dots,d_n)\}) = V(M', \bar{q}x_i\ \beta[d_1,\dots,d_n])$

This proves the validity of (9.2). In the same manner one proceeds if $\bar{q}\varepsilon\bar{U}$.

Proposition 9.2 (Lowenheim-Skolem Theorem) Let $V = (V, I)$ be a valuation structure with definable quantifiers and $\tau : L \to V$ a V-system of axioms with an infinite completely valued model M . Then, for every cardinal $\gamma \geqslant |L| + p(M)$, there exists a completely valued model M' of τ of cardinality γ . If M is with classical equality, then M' is with classical equality, too.

Proof. Let I be a set and I a ultrafilter on I such that the cardinality of the ultrapower D^I/I is greater than γ . Let M^* the ultrapower of M via I . Then from Proposition 8.3 it follows that M^* is a model of τ .

Let X be a subset of M^* of cardinality γ , then Proposition 9.1 assures that there exists an elementary submodel of M^* of cardinality γ containing X. This is the desired model.

Recall that even in Chang and Keisler's continuous logic [2] a ultra product concept is given and, consequently, a Lowenheim-Skolem theorem is proved. But, is continuous logic, we are forced to assume that connectives and quantifiers are continuous with respect to a suitable Hausdorff topology. This excludes, for example, the Lindenbaum or the Brouwerian algebras as valuation structures. In this sense Proposition 9.2 is more general even if Chang and Keisler's result assures that V(M')=V(M) and not merely that V(M') is an elementary extension of V(M).

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